Generalized Circulants and Class Functions of Finite Groups

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Dedicated to Bertram Huppert on his 60th birthday.

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ABSTRACT

The regular representation of the ring of class functions of a finite group is used to construct families of matrices that enjoy many of the properties of circulant matrices. For different choices of finite groups one gets different families of matrices: the circulants, block circulants, block circulants with circulant blocks of all levels, and many others. The properties of these families are simple consequences of the character theory of finite groups. Many known properties of circulants and some of their generalizations are obtained as special cases of the properties of the families constructed. Some applications to character theory are also mentioned.

INTRODUCTION

Let G be a finite group, and let $\chi_1, \chi_2, \dots, \chi_k$ be a fixed enumeration of the irreducible complex characters of G. For every complex class function θ of G, let $M(\theta)$ be the $k \times k$ matrix whose (i, j)th entry is $[\theta\chi_i, \chi_j]$, where $[\ ,\ ]$ is the usual inner product defined in the algebra, cf(G), of all complex class functions of G. The purpose of this article is to study the family of matrices $M(G) = \{M(\theta) \mid \theta \in \mathrm{cf}(G)\}$. Each $M(\theta)$ is called a G-circulant, and $M(G)$ is the set of G-circulants. The reason for the name is that $M(G)$ enjoys many of the properties of the regular circulant matrices and some of

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their generalizations. For different choices of G (and different choices of enumerations of the irreducible characters of G) one gets different families of G-circulants. If G is a cyclic group of order n, then (using the usual enumeration of the irreducible characters of G) $M(G)$ is, in fact, the set of all $n \times n$ circulants. The set of all block circulants of level r and type $(n_1, n_2, \dots, n_r)$ (see p. 188 of [7] or our Section 3 for definition) is a set of G-circulants for the group $G = C_{n_1} \times C_{n_2} \times \cdots \times C_{n_r}$, where C_{n_i} is a cyclic group of order n_i. Details on these examples and others can be found in Section 3.

In Section 1 we describe the properties of $M(G)$. Among other things, $M(G)$ is a commutative algebra over the complex number field $\mathbb{C}$. Also $M(G)$ is isomorphic to $\mathrm{cg}(G)$, and $M(G)$ is closed under M-P inversion and also under the * (conjugate-transpose) operator. Moreover, each $M(\theta) \in M(G)$ is a normal matrix which is diagonalized by the character table matrix of G. Formulas for the eigenvalues of the matrices in $M(G)$ are given, as well as some characterizations of $M(G)$ and other properties. The G-circulants corresponding to $G = H \times K$, a direct product of the two finite subgroups H and K, are described in Section 2. Each such matrix is a linear combination of Kronecker products of matrices corresponding to H and K. In fact, a matrix A is a $H \times K$-circulant if and only if A is a block H-circulant in which the blocks are K-circulants. This will enable us to construct new examples of G-circulants from old ones. Also in Section 2 are some details on block G-circulants and block matrices with G-circulant blocks, which generalize known results on block circulants and on block matrices with circulant blocks.

In the example section, 3, we indicate why the properties of Sections 1 and 2 are, in fact, generalizations of the properties of circulants and some of their generalizations. We show that many known properties of circulants (and some of their generalizations) follow from the general properties of G-circulants as special cases.

Finally, in Section 4, we give a very brief description on how the isomorphism $M(G) \cong \mathrm{cg}(G)$ and the theory of nonnegative matrices can be used to obtain results on characters of finite groups. For more details on such applications see [1], [5], and [6].

In [3], [4], and [10] finite groups are used to generalize circulants. In most of these places G is assumed to be abelian, in which case G is isomorphic to the sets of its irreducible characters and some of the generalized circulants obtained are similar to ours. For nonabelian groups ([4] and one of [10]) the sets of matrices are different from ours.

Our basic notation is standard. The group-theoretical notation and notions are taken mainly from the first four chapters of [8], and those on matrices from [7] and Chapter 2 of [2].

1. G-CIRCULANTS AND THEIR PROPERTIES

For a *finite group* G, we denote by $\mathrm{Irr}(G)$ the set of all irreducible complex characters of G and by $\mathrm{cf}(G)$ the set of all complex class functions of G. The set $\mathrm{cf}(G)$ is known to be a $\mathbb{C}$-algebra, where $\mathbb{C}$ is the field of complex numbers. Moreover, $\mathrm{Irr}(G)$ is an orthonormal basis of $\mathrm{cf}(G)$ with respect to a natural inner product, $[\ ,\ ]$, defined on $\mathrm{cf}(G)$. See [8] for more details. Let $C_1, C_2, \ldots, C_k$ be all the conjugacy classes of G. If $\theta \in \mathrm{cf}(G)$, then θ is completely determined by its values on the C_i's. Let $\alpha = (\chi_1, \chi_2, \ldots, \chi_k)$ be an ordering of $\mathrm{Irr}(G)$, namely, $\chi_1, \chi_2, \ldots, \chi_k$ are all the irreducible complex characters of G in this specific enumeration.

For each complex class function, θ, of G define $M^\alpha(\theta)$ to be the $k \times k$ matrix $M^\alpha(\theta) = m^\alpha_{ij}(\theta))$, where $m^\alpha_{ij}(\theta) = [\theta\chi_i, \chi_j]$, $1 \leqslant i, j \leqslant k$. We call each $M^\alpha(\theta)$ a (G, α)-circulant. Set $M(G, \alpha) = \{M^\alpha(\theta) \mid \theta \in \mathrm{cf}(G)\}$. Another matrix that will be used is the *character table* matrix $X(G, \alpha)$ of G, i.e., $X(G, \alpha) = (\chi_i(C_j))$, where $1 \leqslant i, j \leqslant k$. We denote by $\mathrm{diag}(a_1, a_2, \ldots, a_n)$ the diagonal matrix whose main diagonal entries are $a_1, a_2, \ldots, a_n$.

Note that $\theta\chi_i = \sum_{j=1}^k m^\alpha_{ij}(\theta)\chi_j$ for each $\theta \in \mathrm{cf}(G)$. It follows that if α' is another ordering of $\mathrm{Irr}(G)$, then $M^{\alpha'}(\theta)$ is permutationally similar to $M^\alpha(\theta)$. In fact the following is obvious.

LEMMA 1.1. *Let G be a finite group, and α an ordering of* $\mathrm{Irr}(G)$. *Set* $d = |\mathrm{Irr}(G)|$.

(a) *If α' is another ordering of* $\mathrm{Irr}(G)$, *then there exists a permutation matrix P such that $P^{-1}M^\alpha(\theta)P = M^{\alpha'}(\theta)$ for all $\theta \in \mathrm{cf}(G)$.*

(b) *If Q is any $d \times d$ permutation matrix, then there exists an ordering α' of* $\mathrm{Irr}(G)$ *such that $M^{\alpha'}(\theta) = Q^{-1}M^\alpha(\theta)Q$ for all $\theta \in \mathrm{cf}(G)$.*

Next we list several simple properties of the mapping $\theta \to M^\alpha(\theta)$.

PROPOSITION 1.2. *Let G be a finite group, and α an ordering of* $\mathrm{Irr}(G)$. *Then*:

(a) *The mapping $\theta \to M^\alpha(\theta)$ is a $\mathbb{C}$-linear, additive, multiplicative and one-to-one mapping from* $\mathrm{cf}(G)$ *onto $M(G, \alpha)$.*

(b) *$M^\alpha(\bar{\theta}) = (M^\alpha(\theta))^*$ for all $\theta \in \mathrm{cf}(G)$ (here $\bar{\theta}$ is the complex conjugate class function defined by $\bar{\theta}(g) = \overline{\theta(g)}$ for all $g \in G$).*

(c) *$M^\alpha(1_G) = I$, where I is the identity matrix and 1_G is the principal character of G.*

Proof. Set $\alpha = (\chi_1, \chi_2, \ldots, \chi_k)$, $M(\theta) = M^{\alpha}(\theta)$, and $m_{ij}(\theta) = m^{\alpha}_{ij}(\theta)$ for all $\theta \in \mathrm{cf}(G)$ and $1 \leqslant i, j \leqslant k$. Let $\theta, \eta \in \mathrm{cf}(G)$. Then $\theta\chi_i = \sum_{l=1}^{k} m_{il}(\theta)\cdot\chi_l$ and $\eta\chi_l = \sum_{j=1}^{k} m_{lj}(\eta)\cdot\chi_j$. Hence $\theta\eta\chi_i = \sum_{l=1}^{k} m_{il}(\theta)\cdot\eta\chi_l = \sum_{l=1}^{k}\sum_{j=1}^{k} m_{il}(\theta)m_{lj}(\eta)\cdot\chi_j = \sum_{j=1}^{k}\{\sum_{l=1}^{k} m_{il}(\theta)m_{lj}(\eta)\}\chi_j$. It follows that $[\theta\eta\cdot\chi_i, \chi_j] = \sum_{l=1}^{k} m_{il}(\theta)m_{lj}(\eta)$ and so $M(\theta\eta) = M(\theta)M(\eta)$. Since $[(\theta+\eta)\chi_i, \chi_j] = [\theta\chi_i, \chi_j] + [\eta\chi_i, \chi_j]$, we get that $M(\theta+\eta) = M(\theta) + M(\eta)$, and as $[\lambda\theta\chi_i, \chi_j] = \lambda[\theta\chi_i, \chi_j]$ for all $\lambda \in \mathbb{C}$, we have that $M(\lambda\theta) = \lambda M(\theta)$ for all $\lambda \in \mathbb{C}$. Next,

$$\left[\bar{\theta}\chi_i, \chi_j\right] = (1/|G|)\sum_{g\in G} \overline{\theta(g)}\,\chi_i(g)\,\overline{\chi_j(g)}$$

$$= (1/|G|)\sum_{g\in G} \chi_i(g)\cdot\overline{\theta(g)\chi_j(g)} = \left[\chi_i, \theta\chi_j\right],$$

where $|G|$ is the order of G (see [8, p. 20]). Thus, $[\bar{\theta}\chi_i, \chi_j] = [\chi_i, \theta\chi_j] = \overline{\left[\theta\chi_j, \chi_i\right]}$ and so $M(\bar{\theta}) = \overline{(M(\theta))}^{t} = (M(\theta))^{*}$.

Next, to see that the mapping is one-to-one, assume that $M(\theta) = M(\eta)$ and let r be the index such that $\chi_r = 1_G$, the principal character of G. Then $\theta = \theta\chi_r = \sum_{l=1}^{k} m_{rl}(\theta)\chi_l = \sum_{l=1}^{k} m_{rl}(\eta)\chi_l = \eta\chi_r = \eta$, as desired. By the definition of $M(G,\alpha)$, the mapping is onto. Finally $M^{\alpha}(1_G) = ([1_G\chi_i, \chi_j]) = ([\chi_i, \chi_j]) = I$. ∎

The rest of this section is devoted to listing and proving the main properties of (G,α)-circulants. In view of the examples of Section 3, all the following results of [7] are special cases of these properties: all the results of pp. 67, 68, 72, 73; Theorem 3.3.1 and its corollary; Theorem 3.3.8: Theorems 5.8.1, 5.8.2, 5.8.3, 5.8.4 with their corollary; Theorem 5.8.5; all the results on circulants and block circulants with circulant blocks of Chapter 6.5; some of the results of Chapters 3.4 and 3.6; several problems (e.g. Problems 18, 19 of p. 81 and 10 of p. 70); and others. See Section 3 for details.

COROLLARY 1.3. *Let G be a finite group, and α an ordering of* Irr(G). *Then $M(G,\alpha)$ is a commutative algebra over $\mathbb{C}$, and the mapping $\theta \to M^{\alpha}(\theta)$ is an algebra isomorphism from* cf(G) *onto $M(G,\alpha)$. Moreover (and in particular), the following are true for any two (G,α)-circulants A and B:*

(a) *AB, A^{*}, A^{t} and $\alpha_1 A + \alpha_2 B$ are (G,α)-circulants for any $\alpha_1, \alpha_2 \in \mathbb{C}$. Also the M-P inverse of A is a (G,α)-circulant.*

(b) *$AB = BA$.*

(c) *A is a normal matrix.*

Proof. Set $k = |\mathrm{Irr}(G)|$. Then by Proposition 1.2 $M(G,\alpha)$ is the image of the one-to-one algebra homormophism $\theta \to M^{\alpha}(\theta)$ from cf(G) to the algebra of all $k \times k$ matrices over $\mathbf{C}$. Hence, $M(G,\alpha)$ is a $\mathbf{C}$-algebra, and $M(G,\alpha) \cong \mathrm{cf}(G)$ as algebras. Since $\mathrm{cf}(G)$ is commutative, so is $M(G,\alpha)$. This proves all assertions except the ones with A^*, A^t, the M-P inverse, and (c). By Proposition 1.2, part (b), if $A = M^\alpha(\theta)$ then $A^* = M^\alpha(\bar{\theta}) \in M(G,\alpha)$, and as elements of $M(G,\alpha)$ commute we get also that $AA^* = A^*A$, as desired. Set $\theta = \sum_i a_i \chi_i$, $\eta = \sum_i a_i \bar{\chi}_i$, where $a_i \in \mathbf{C}$ and $\chi_i \in \mathrm{Irr}(G)$. Then $(M^\alpha(\theta))^t = M^\alpha(\eta)$, as $(M^\alpha(\chi_i))^* = (M^\alpha(\chi_i))^t$ for all i. Theorem 6.3.2 of [7] implies that if $A \in M(G,\alpha)$ then the M-P inverse of A is in $M(G,\alpha)$. ∎

Remark. A more elementary demonstration that $M(G,\alpha)$ is closed under M-P inversion will be given later, by giving the explicit form of the M-P inverse of (G,α)-circulants.

The next theorem contains some key observations concerning (G,α)-circulants. Although a proof of its main parts (a,b) appears in [5], we will give here a (different) proof, for the sake of self-containment of this article.

Theorem 1.4. *Let G be a finite group, and α an ordering of* $\mathrm{Irr}(G)$ *such that* $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \dots, \chi_k\}$. *Denote by $C_1, C_2, \dots, C_k$ the conjugacy classes of G, and by $X = X(G,\alpha)$ the character table of G. Then*:

(a) *The columns of X are a universal set of (right) eigenvectors for all (G,α)-circulants.*

(b) *For each (G,α)-circulant, $M^\alpha(\theta)$, we have that $X^{-1}M^\alpha(\theta)X = \mathrm{diag}(\theta(C_1), \theta(C_2), \dots, \theta(C_k))$.*

(c) *If $D = (1/|G|)\mathrm{diag}(|C_1|, |C_2|, \dots, |C_k|)$ then $XDX^* = I$. In particular, if G is abelian, $(1/\sqrt{|G|})X$ is unitary.*

(d) *The set $\{M^\alpha(\chi_i) \mid 1 \leqslant i \leqslant k\}$ is a basis of $M(G,\alpha)$. Furthermore, if $M = \sum_{i=1}^k a_i M^\alpha(\chi_i)$ is any (G,α)-circulant, then the eigenvalues of M are $\sum_{i=1}^k a_i \chi_i(C_j)$ for $j = 1, 2, \dots, k$.*

(e) *The minimal polynomial of the (G,α)-circulant $M^\alpha(\theta)$ is $\prod_{i=1}^m (x - \alpha_i)$, where the α_i are the distinct values taken on by θ. The characteristic polynomial of $M^\alpha(\theta)$ is $\prod_{i=1}^k \{x - \theta(C_i)\}$ (here x is the variable).*

(f) *Let D be as in* (c), *and $U = XD(XD)^*$. For any two (G,α)-circulants A and B define $[A,B] = \mathrm{tr}(AUB^*)$. Then $[\ ,\ ]$ is an inner product on $M(G,\alpha)$ and $\theta \to M^\alpha(\theta)$ is an isometry, namely $[\theta,\eta] = [M^\alpha(\theta), M^\alpha(\eta)]$ for all $\theta, \eta \in \mathrm{cf}(G)$. (Note that if G is abelian then $U = (1/|G|)I$.)*

Proof. (a), (b), (c), (e): The notation $(w)_i$ will be used to denote the ith coordinate of the vector w. Let v^r be the rth column of X. Then $v^r = (\chi_1(C_r), \chi_2(C_r), \dots, \chi_k(C_r))^t$. We will show that v^r is an eigenvector of $M^\alpha(\theta)$ corresponding to the eigenvalue $\theta(C_r)$. Namely, we will show that

$M^{\alpha}(\theta)v^{r}=\theta(C_r)v^{r}$. Set $M^{\alpha}(\theta)=(m_{ij})$. For each i we have that

$$\begin{aligned}(M^{\alpha}(\theta)v^{r})_i &= \sum_{j=1}^{k} m_{ij}(v^{r})_j=\sum_{j=1}^{k}\left[\theta\chi_i,\chi_j\right]\chi_j(C_r)\\ &=\frac{1}{|G|}\sum_{g\in G}\sum_{j=1}^{k}\theta(g)\chi_i(g)\overline{\chi_j(g)}\,\chi_j(C_r)\\ &=\frac{1}{|G|}\sum_{g\in G}\theta(g)\chi_i(g)n(g),\end{aligned}$$

where (using the second orthogonality relations of characters)

$$n(g)=\sum_{j=1}^{k}\chi_j(C_r)\overline{\chi_j(g)}=\begin{cases}0 & \text{if } g\notin C_r,\\ |G|/|C_r| & \text{if } g\in C_r.\end{cases}$$

Therefore

$$\begin{aligned}(M^{\alpha}(\theta)v^{r})_i &= \frac{1}{|G|}\sum_{g\in C_r}\theta(g)n(g)\chi_i(g)\\ &=\frac{1}{|G|}\cdot|C_r|\theta(C_r)\chi_i(C_r)\frac{|G|}{|C_r|}\\ &=\theta(C_r)\chi_i(C_r)=\theta(C_r)(v^{r})_i,\end{aligned}$$

as claimed.

Now part (c) is well known (e.g., see [8, p. 22]), and it implies that X is nonsingular. Hence the above argument shows that the columns of X are k linearly independent eigenvectors of $M^{\alpha}(\theta)$. Hence $X^{-1}M^{\alpha}(\theta)X=\operatorname{diag}(\theta(C_1),\theta(C_2),\dots,\theta(C_k)$. So (a), (b) and (c), are proved. Since $M^{\alpha}(\theta)$ is diagonalizable, part (e) follows.

(d): As $\operatorname{Irr}(G)$ is a basis of $\operatorname{cf}(G)$, the set $\{M^{\alpha}(\chi_i)\,|\,1\leqslant i\leqslant k\}$ is a basis of $M(G,\alpha)$ by Corollary 1.3. By part (b) we get that

$$\begin{aligned}X^{-1}MX &= \sum_{i=1}^{k}a_iX^{-1}M^{\alpha}(\chi_i)X=\sum_{i=1}^{k}a_i\operatorname{diag}(\chi_i(C_1),\dots,\chi_i(C_k))\\ &=\operatorname{diag}\left(\sum_{i=1}^{k}a_i\chi_i(C_1),\sum_{i=1}^{k}a_i\chi_i(C_2),\dots,\sum_{i=1}^{k}a_i\chi_i(C_k)\right),\end{aligned}$$

as desired.

(f): We have that

$$[\theta,\eta] = \frac{1}{|G|}\sum_{g\in G}\theta(g)\overline{\eta(g)} = \sum_{i=1}^{k}\theta(C_i)\frac{|C_i|}{|G|}\overline{\eta(C_i)} = \mathrm{tr}(A),$$

where

$$A = \mathrm{diag}(\theta(C_1),\theta(C_2),\dots,\theta(C_k))\cdot D\cdot \mathrm{diag}(\overline{\eta(C_1)},\overline{\eta(C_2)},\dots,\overline{\eta(C_k)}).$$

Now

$$\mathrm{tr}(A) = \mathrm{tr}(XAX^{-1}) = \mathrm{tr}(M^{\alpha}(\theta)\cdot XDX^{-1}\cdot(M^{\alpha}(\eta))^{*}).$$

Note that $X^{-1} = DX^{*}$ and $D = D^{*}$. Thus $XDX^{-1} = XDDX^{*} = XD(XD)^{*} = U$ and consequently $[\theta,\eta] = \mathrm{tr}(M^{\alpha}(\theta)U(M^{\alpha}(\eta))^{*})$ as needed. ■

THEOREM 1.5. *Let G be a finite group and α an ordering of* $\mathrm{Irr}(G)$. *Set $k = |\mathrm{Irr}(G)|$ and $X = X(G,\alpha)$. Then*

(1) *the following are equivalent for a $k\times k$ complex matrix A:*
   (i) *A is a (G,α)-circulant;*
   (ii) *A commutes with $M^{\alpha}(\chi)$ for all $\chi\in\mathrm{Irr}(G)$;*
   (iii) *$A = X\Delta X^{-1}$ for some diagonal complex matrix Δ.*

(2) *If $\theta\in\mathrm{cf}(G)$ has k distinct values, then $M(G,\alpha)$ is the set of all polynomials in $M(\theta)$ with complex coefficients.*

Proof. (i) $\Leftrightarrow$ (ii): If A is a (G,α)-circulant, it commutes with all (G,α)-circulants and in particular with any $M^{\alpha}(\chi)$ for $\chi\in\mathrm{Irr}(G)$. Conversely, assume that A commutes with each $M^{\alpha}(\chi)$, $\chi\in\mathrm{Irr}(G)$. Let $b_1,b_2,\dots,b_k$ be distinct complex numbers. Set $\mathrm{Irr}(G)=\{\chi_1,\chi_2,\dots,\chi_k\}$ (according to α), and let $C_1,C_2,\dots,C_k$ be all the conjugacy classes of G. Define a class function θ on G by $\theta(C_i)=b_i$ for all $i=1,2,\dots,k$. Then, by the previous theorem, the matrix $M^{\alpha}(\theta)$ has k distinct eigenvalues: $b_1,b_2,\dots,b_k$. Also, for some complex numbers $a_1,a_2,\dots,a_k$ we have that $M^{\alpha}(\theta)=\sum_{i=1}^{k}a_iM^{\alpha}(\chi_i)$, and as A commutes with each of the summands, A commutes with $M^{\alpha}(\theta)$. Now, the minimal polynomial of $M^{\alpha}(\theta)$ coincides with the characteristic polynomial of $M^{\alpha}(\theta)$, and it is well known that then A is a polynomial in $M^{\alpha}(\theta)$ (see for example [7, Theorem 6.A.2]). As $M^{\alpha}(\theta)$ belongs to the algebra $M(G,\alpha)$, so does A. This also proves (2).

(i) $\Leftrightarrow$ (iii): Let D be the set of all diagonal complex $k\times k$ matrices. Then $XDX^{-1}=\{X\Delta X^{-1}\mid\Delta\in D\}$ is a $\mathbb{C}$-vector space of dimension k. As $M(G,\alpha)\subseteq XDX^{-1}$ and $M(G,\alpha)$ is of dimensional k (by Theorem 1.4), we get equality. ■

Notation. Following [7], we define, for every $\lambda \in \mathbb{C}$, the number $\lambda^{\div}$ by

$$\lambda^{\div} = \begin{cases} 1/\lambda & \text{if } \lambda \neq 0, \\ 0 & \text{if } \lambda = 0. \end{cases}$$

If $\Delta = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k)$, we set $\Delta^{\div} = \operatorname{diag}(\lambda_1^{\div}, \lambda_2^{\div}, \ldots, \lambda_k^{\div})$.

The next corollaries are all consequences of the simultaneous diagonalization of the (G, α)-circulants by $X(G, \alpha)$ (compare with the analogous statements on circulants in [7, pp. 81, 87, 102, 103]).

Corollary 1.6. *Let G be a finite group and α an ordering of* $\operatorname{Irr}(G)$. *Set $X = X(G, \alpha)$ and let A be an arbitrary (G, α)-circulant. Then*:

(a) *Set $A = X\Delta X^{-1}$, where $\Delta = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k)$. If $A^{\div}$ is the M-P inverse of A, then $A^{\div} = X\Delta^{\div}X^{-1} = \sum_{i=1}^{k}\lambda_i^{\div}B_i$, where $B_i = X\Delta_i X^{-1}$, $\Delta_i =$* $\operatorname{diag}(0, \ldots, 0, 1, 0, \ldots, 0)$ *(the 1 occurs in the ith place), $i = 1, 2, \ldots, k$. Also $B_i^{\div} = B_i$ and $A = \sum_{i=1}^{k}\lambda_i B_i$ (the spectral decomposition of A).*

(b) *A is Hermitian if and only if its eigenvalues are all real.*

(c) *A is unitary if and only if its eigenvalues lie on the unit circle.*

Proof. (a): Let $T = (1/\sqrt{|G|})\operatorname{diag}(\sqrt{|C_1|}, \sqrt{|C_2|}, \ldots, \sqrt{|C_k|})$, where the C_i's are conjugacy classes of G. Then $T^2 = D$ of Theorem 1.4. Set $Y = XT$; then by the equality $XDX^* = I$ we get that $YY^* = I$. Clearly $A = X\Delta X^{-1} = Y\Delta Y^{-1}$ and $X\Delta^{\div}X^{-1} = Y\Delta^{\div}Y^{-1}$. Now the fact that $A^{\div} = Y\Delta^{\div}Y^{-1}$ follows from the definition of $A^{\div}$ (see p. 44 of [7]). The rest of the statement is obvious.

(b) and (c) hold for all normal matrices; in particular they hold for A. ∎

Corollary 1.7. *Let G be a finite group and α an ordering of* $\operatorname{Irr}(G)$, *then all the corollaries of pp.* 102, 103 *of* [7] *still hold if we replace everywhere the word "circulant" by "(G, α)-circulants".*

Proof. As in pp. 102, 103 of [7]. ∎

2. DIRECT PRODUCTS, BLOCK (G, α)-CIRCULANTS, AND BLOCK MATRICES WITH (G, α)-CIRCULANT BLOCKS

Notation. Let $G = H \times K$ be a finite group which is the direct product of its two subgroups, H and K. For each $\eta \in \operatorname{cf}(H)$ and $\theta \in \operatorname{cf}(K)$ define a class function $\eta \times \theta$ of G by $(\eta \times \theta)(xy) = \eta(x)\theta(y)$ for all $x \in H$, $y \in K$. It is well known that $\mathrm{Irr}(G) = \{\eta \times \theta \mid \eta \in \mathrm{Irr}(H),\ \theta \in \mathrm{Irr}(K)\}$ (see p. 59 of [8]). Next, suppose that M_1 and M_2 are two $\mathbb{C}$-algebras of matrices. Let $\{A_1, A_2, \ldots, A_m\}$ be a basis of M_1, and $\{B_1, B_2, \ldots, B_r\}$ a basis of M_2. The Kronecker product $M_1 \otimes M_2$ of M_1 and M_2 is defined to be the space spanned by the set $\{A_i \otimes B_j \mid 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant r\}$, where $\otimes$ means the Kronecker product. It is obvious that $M_1 \otimes M_2$ is independent of the bases chosen for M_1 and M_2. In fact $M_1 \otimes M_2$ is the set of all matrices of the form $\sum_{i=1}^{u} \sum_{j=1}^{v} a_{ij}(R_i \otimes S_j)$ where $a_{ij} \in \mathbb{C}$, $R_i \in M_1$, $S_j \in M_2$, and u, v are positive integers. Moreover, as $a_{ij}(R_i \otimes S_j) = (a_{ij}R_i \otimes S_j)$ and $R_i \in M_1$ if and only if $a_{ij}R_i \in M_1$, we get that $M_1 \otimes M_2$ is the set of all matrices of the form $\sum_{i=1}^{u} \sum_{j=1}^{v} C_i \otimes D_j$ where $C_i \in M_1$ and $D_j \in M_2$, $1 \leqslant i \leqslant u$, $1 \leqslant j \leqslant v$, and u, v are positive integers.

THEOREM 2.1. *Let G be a finite group such that $G = H \times K$, where H and K are subgroups of G. Suppose that β is an ordering of* $\mathrm{Irr}(H)$ *and γ is an ordering of* $\mathrm{Irr}(K)$. *Then there exists an ordering α of* $\mathrm{Irr}(G)$ *such that*

(a) $M^{\alpha}(\eta \times \theta) = M^{\beta}(\eta) \otimes M^{\gamma}(\theta)$ *for each* $\eta \in \mathrm{cf}(H)$ *and* $\theta \in \mathrm{cf}(K)$;
(b) $M(G, \alpha) = M(H, \beta) \otimes M(K, \gamma)$;
(c) $X(G, \alpha) = X(H, \beta) \otimes X(K, \gamma)$.

Proof. According to β and γ, set $\mathrm{Irr}(H) = \{\eta_1, \eta_2, \ldots, \eta_r\}$ and $\mathrm{Irr}(K) = (\theta_1, \theta_2, \ldots, \theta_s\}$. Let $\chi_{(a,b)} = \eta_a \times \theta_b$ for $1 \leqslant a \leqslant r$, $1 \leqslant b \leqslant s$. Let α be the following ordering of $\mathrm{Irr}(G)$:

$$\big(\chi_{(1,1)}, \chi_{(1,2)} \ldots, \chi_{(1,s)}, \chi_{(2,1)}, \chi_{(2,2)}, \ldots, \chi_{(2,s)}, \ldots,$$

$$\chi_{(a,1)}, \chi_{(a,2)}, \ldots, \chi_{(a,s)}, \ldots, \chi_{(r,1)}, \chi_{(r,2)}, \ldots, \chi_{(r,s)}\big).$$

Note that the indices (a, b) in the ordering α are ordered lexicographically.

(a): Let $\eta \in \mathrm{cf}(H)$ and $\theta \in \mathrm{cf}(K)$. Then, $\eta \times \theta = (\eta \times 1_K)(1_H \times \theta)$, so that $M^{\alpha}(\eta \times \theta) = M^{\alpha}(\eta \times 1_K) \cdot M^{\alpha}(1_H \times \theta)$. If we prove that $M^{\alpha}(\eta \times 1_K) = M^{\beta}(\eta) \otimes M^{\gamma}(1_K)$ and $M^{\alpha}(1_H \times \theta) = M^{\beta}(1_H) \otimes M^{\gamma}(\theta)$, then by Proposition 1.2 (c) and [7, p. 23] we will get that $M^{\alpha}(\eta \times \theta) = (M^{\beta}(\eta) \otimes I_s)(I_r \otimes M^{\gamma}(\theta)) = M^{\beta}(\eta) \otimes M^{\gamma}(\theta)$, as desired (here I_n is the $n \times n$ identity matrix). Hence, in order to prove (a) it suffices to show that $M^{\alpha}(\eta \times 1_K) = M^{\beta}(\eta) \otimes I_s$ and $M^{\alpha}(1_H \times \theta) = I_r \otimes M^{\gamma}(\theta)$. Write $M^{\beta}(\eta) = (m_{ij})$, $1 \leqslant i, j \leqslant r$, and $M^{\gamma}(\theta) = (n_{lt})$, $1 \leqslant l, t \leqslant s$. Then for each a, $1 \leqslant a \leqslant r$, and each b, $1 \leqslant b \leqslant s$, we have that $\eta\eta_a = \sum_{i=1}^{r} m_{ai}\eta_i$ and $\theta\theta_b = \sum_{i=1}^{s} n_{bi}\theta_i$.

Each of the matrices $M^{\alpha}(\eta \times 1_K)$ and $M^{\alpha}(1_H \times \theta)$ is an $rs \times rs$ matrix. We label the rows and columns according to the ordering α and write the

TABLE 1

	$(1,1),(1,2),\ldots,(1,s)$	$(2,1),(2,2),\ldots,(2,s)$	$\cdots$	$(d,1),(d,2),\ldots,(d,s)$	$\cdots$	$(r,1),(r,2),\ldots,(r,s)$
$(1,1)$ $(1,2)$ $\cdots$ $(1,s)$	A_{11}	A_{12}	$\cdots$	A_{1d}	$\cdots$	A_{1r}
$(2,1)$ $(2,2)$ $\cdots$ $(2,s)$	A_{21}	A_{22}	$\cdots$	A_{2d}	$\cdots$	A_{2r}
$\cdots$	$\cdots$	$\cdots$		$\cdots$		$\cdots$
$(a,1)$ $(a,2)$ $\cdots$ (a,s)	A_{a1}	A_{a2}	$\cdots$	A_{ad}	$\cdots$	A_{ar}
$\cdots$	$\cdots$	$\cdots$		$\cdots$		$\cdots$
$(r,1)$ $(r,2)$ $\cdots$ (r,s)	A_{r1}	A_{r2}	$\cdots$	A_{rd}	$\cdots$	A_{rr}

matrices as block matrices as shown in Table 1, where the A_{ij}'s are $s \times s$ matrices.

Consider $M^{\alpha}(\eta \times 1_k)$ first. The entries of its (a, b)th row are the coefficients of the $\chi_{(u,v)}$'s in the expansion of $(\eta \times 1_K)\chi_{(a,b)}$, ordered according to α. Now, $(\eta \times 1_k)\chi_{(a,b)} = (\eta \times 1_k)(\eta_a \times \theta_b) = \eta\eta_a \times \theta_b = (\sum_{i=1}^{r} m_{ai}\eta_i) \times \theta_b$ and hence $(\eta \times 1_k)\chi_{(a,b)} = \sum_{i=1}^{r} m_{ai}\chi_{(i,b)}$. Note that the entries of the bth row of A_{ad} are the coefficients of $\chi_{(d,1)}, \chi_{(d,2)}, \dots, \chi_{(d,s)}$ in the above expansion. Of these, the coefficient of $\chi_{(d,b)}$ is m_{ad} (regardless of what b is), and the coefficients of $\chi_{(d,i)}$, $i \neq b$, are all zeros. Thus $A_{ad} = m_{ad}I_s$, and consequently $M^{\alpha}(\eta \times 1_k) = M^{\beta}(\eta) \otimes I_s$, as claimed.

Similarly, the entries of the (a, b)th row of $M^{\alpha}(1_H \times \theta)$ are the coefficients of the $\chi_{(u,v)}$'s (ordered according to α) in the expansion of $(1_H \times \theta)\chi_{(a,b)} = (1_H \times \theta)(\eta_a \times \theta_b) = \eta_a \times \theta\theta_b = \sum_{i=1}^{s} n_{bi}\eta_a \times \theta_i$. Now, $(1_H \times \theta)\chi_{(a,b)} = \sum_{i=1}^{s} n_{bi}\chi_{(a,i)}$. Again, the entries of the bth row of A_{ad} are the coefficients in this last expansion of $\chi_{(d,1)}, \chi_{(d,2)}, \dots, \chi_{(d,s)}$. If $a \neq d$, then all the coefficients are zero for all b. Hence, $A_{ad} = 0$ if $a \neq d$. If $a = d$, the ith entry of the bth row of A_{ad} is n_{bi} (regardless of a). We conclude that $A_{aa} = (n_{bi}) = M^{\gamma}(\theta)$ for all a, $1 \leqslant a \leqslant r$, and consequently $M^{\alpha}(1_H \times \theta) = I_r \otimes M^{\gamma}(\theta)$, as desired.

(b): As $\operatorname{Irr}(G) = \{\eta_i \times \theta_j | 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s\}$, we get from Theorem 1.4(d) that $\{M^{\alpha}(\eta_i \times \theta_j) | 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s\}$ is a basis for $M(G, \alpha)$. By part (a), $M(G, \alpha)$ is linearly spanned by the set

$$\left\{M^{\beta}(\eta_i) \otimes M^{\gamma}(\theta_j) \middle| 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s\right\}.$$

Since the sets $\{M^{\beta}(\eta_i) | 1 \leqslant i \leqslant r\}$ and $\{M^{\gamma}(\theta_j) | 1 \leqslant j \leqslant s\}$ are bases for the algebras $M(H, \beta)$ and $M(K, \gamma)$, respectively, we get (b).

(c): Denote by $C_1, C_2, \dots, C_r$ the conjugacy classes of H, and by $D_1, D_2, \dots, D_s$ the conjugacy classes of K. Then the conjugacy classes of G are the sets $C_i \times D_j$, $1 \leqslant i \leqslant r$, $i \leqslant j \leqslant s$. Then $X(G, \alpha) = \left(\chi_{(a,b)}(C_i \times D_j)\right) = \left(\eta_a(C_i)\theta_b(D_j)\right)$, where the $\chi_{(a,b)}$'s and the $C_i \times D_j$'s are ordered according to α. Again this is an $rs \times rs$ matrix, and when arranged as an $r \times r$ block matrix it has the form of Table 1. Then for a fixed (a, d) we have that

$$A_{ad} = \left(\eta_a(C_d)\theta_b(D_j)\right), \qquad 1 \leqslant b, j \leqslant s,$$

and so $A_{ad} = \eta_a(C_d)X(K, \gamma)$, and (c) follows. ∎

We seek for a "better" description of $M(G, \alpha)$ when $G = H \times K$. In fact, using the notation of Theorem 2.1, we will show that $M(G, \alpha)$ is exactly the set of block (H, β)-circulants in which the blocks are (K, γ)-circulants. First we have to define a block (H,β)-circulant. This will be done analogously to the definition of block circulants, namely, a block (H,β)-circulant will be a block matrix in which the "relations" between the blocks are "the same" as the "relations" between the entries of an (H,β)-circulant. In the next definitions, notation, and lemma, we will make the preceding paragraph precise.

DEFINITIONS AND NOTATION. Let G be a finite group and α an ordering of $\operatorname{Irr}(G)$ for which $\operatorname{Irr}(G)=\{\chi_1,\chi_2,\dots,\chi_k\}$. The leading index, $m(G,\alpha)$, of the pair (G,α) is the positive integer m such that $\chi_m=1_G$. If A is any $k\times k$ matrix we call the $m(G,\alpha)$th row of A the leading row of A.

If A is a (G,α)-circulant, the leading row of A uniquely determines A. In fact we have:

LEMMA 2.2. *Let G be a finite group and α an ordering of* $\operatorname{Irr}(G)$ *such that* $\operatorname{Irr}(G)=\{\chi_1,\chi_2,\dots,\chi_k\}$. *Let $A=(a_{ij})$ be a $k\times k$ matrix, and let its leading row be $a_m=(a_{m1},a_{m2},\dots,a_{mk})$. Then the following are equivalent:*

(i) *A is a (G,α)-circulant;*
(ii) *$A=M^{\alpha}(\theta)$, where $\theta=\sum_{u=1}^{k}a_{mu}\chi_u$;*
(iii) *$a_{ij}=\sum_{u=1}^{k}a_{mu}[\chi_u\chi_i,\chi_j]$ for $1\leqslant i,j\leqslant k$.*

Proof. If (i) holds, then $A=M^{\alpha}(\theta)$ for some $\theta\in\operatorname{cf}(G)$. As $\theta=\theta 1_G=\theta\cdot\chi_m$, we get that $[\theta,\chi_u]=a_{mu}$ and so (ii) holds. Clearly (ii) implies (iii), as $a_{ij}=[\theta\chi_i,\chi_j]$. Finally, assume that (iii) holds and define a class function $\phi\in\operatorname{cf}(G)$ by $\phi=\sum_{v=1}^{k}a_{mv}\chi_v$. Set $M^{\alpha}(\phi)=(m_{ij})$; then (iii) implies that $a_{ij}=m_{ij}$ for $1\leqslant i,j,\leqslant k$ and so $A=M^{\alpha}(\phi)$ is a (G,α)-circulant. ∎

The "relations" between the entries of a (G,α)-circulant are the k^2 equations of (iii). This motivates:

DEFINITION. Let G be a finite group and α an ordering of $\operatorname{Irr}(G)$ such that $\operatorname{Irr}(G)=\{\chi_1,\chi_2,\dots,\chi_k\}$. Let $A=(A_{ij})$ be a block $k\times k$ matrix where the A_{ij}'s are all square matrices of the same size. Let $A_m=(A_{m1},A_{m2},\dots,A_{mk})$ be the leading row of A. We say that A is a block (G,α)-circulant if and only if $A_{ij}=\sum_{u=1}^{k}A_{mu}[\chi_u\chi_i,\chi_j]$ for $1\leqslant i,j\leqslant k$.

THEOREM 2.3. *Let $G=H\times K$ be a finite group which is a direct product of its subgroups H and K. Suppose that β and γ are ordering of* $\operatorname{Irr}(H)$ *and* $\operatorname{Irr}(K)$, *respectively, and α the ordering of* $\operatorname{Irr}(G)$ *for which $M(G,\alpha)=M(H,\beta)\otimes M(K,\gamma)$. Then a matrix is a (G,α)-circulant if and* *only if it is a block (H,β)-circulant in which the blocks are all (K,γ)-circulants.*

Proof. Let r, s, η_i, and θ_j, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$, be as in the proof of Theorem 2.1. Let A be a (G,α)-circulant. Then, using Theorem 2.1 and the remarks before it, we have that $A = \sum_{u=1}^{l}\sum_{v=1}^{n} M^{\beta}(\tau_u)\otimes M^{\gamma}(\phi_v)$, where $\tau_u \in \mathrm{cf}(H)$ and $\phi_v \in \mathrm{cf}(K)$, $1 \leqslant u \leqslant l$, $1 \leqslant v \leqslant n$, and l and n are positive integers. Set $M^{\beta}(\tau_u) = (m_{ij}(\tau_u))$, $1 \leqslant i, j \leqslant r$. As A is an $rs \times rs$ matrix, we can write it as an $r \times r$ block matrix in which the blocks, which will be denoted by A_{ij}, $1 \leqslant i, j \leqslant r$, are all $s \times s$ matrices. Now

$$A_{ij} = \sum_{u=1}^{l}\sum_{v=1}^{n} m_{ij}(\tau_u) M^{\gamma}(\phi_v). \tag{$*$}$$

Hence, A_{ij} is a linear combination of (K,γ)-circulants, and so each A_{ij} is a (K,γ)-circulant. Next, let $m_w = (m_{w1}(\tau_u), m_{w2}(\tau_u), \dots, m_{wr}(\tau_u))$ be the leading row of $M^{\beta}(\tau_u)$. Then $A_w = (A_{w1}, A_{w2}, \dots, A_{wr})$ is the (block) leading row of A. Since $M^{\beta}(\tau_u)$ is an (H,β)-circulant, it follows from Lemma 2.2 that for all $i, j = 1,2,\dots,r$, we have that $m_{ij}(\tau_u) = \sum_{t=1}^{r} m_{wt}(\tau_u)[\eta_t\eta_i, \eta_j]$. Substituting this in $(*)$ yields

$$A_{ij} = \sum_{t=1}^{r}\left(\sum_{u=1}^{l}\sum_{v=1}^{n} m_{wt}(\tau_u) M^{\gamma}(\phi_v)\right)[\eta_t\eta_i, \eta_j].$$

Again, by $(*)$, we obtain that

$$A_{ij} = \sum_{t=1}^{r} A_{wt}[\eta_t\eta_i, \eta_j]$$

for $1 \leqslant i, j \leqslant r$. This shows that $A = (A_{ij})$ is a block (H,β)-circulant.

Conversely, let B be a block (H,β)-circulant in which the blocks are all (K,γ)-circulants. Then $B = (B_{ij})$, $B_{ij} \in M(K,\gamma)$, $1 \leqslant i, j \leqslant r$. Also, if $B_w = (B_{w1}, B_{w2}, \dots, B_{wr})$ is the leading (block) row of B, then $B_{ij} = \sum_{t=1}^{r} B_{wt}[\eta_t\eta_i, \eta_j]$ for all $i, j = 1,2,\dots,r$. Hence $B = \sum_{t=1}^{r} E_t$, where E_t is the matrix whose (i,j)th (block) entry is $[\eta_t\eta_i, \eta_j]B_{wt}$. As $[\eta_t\eta_i, \eta_j]$ is the (i,j)th entry of $M^{\beta}(\eta_t)$, we get that $E_t = M^{\beta}(\eta_t)\otimes B_{wt}$. Since $M^{\beta}(\eta_t) \in M(H,\beta)$ and $B_{wt} \in M(K,\gamma)$, we conclude that $E_t \in M(G,\alpha)$ and consequently $B = \sum_{t=1}^{r} E_t \in M(G,\alpha)$. ∎

COROLLARY 2.4. *Let $G = G_1 \times G_2 \times \cdots \times G_r$ be a finite group which is a direct product of its subgroups G_i, $1 \leqslant i \leqslant r$. For each $i = 1, 2, \dots, r$ let α_i be an ordering of* $\mathrm{Irr}(G_i)$. *Then for some ordering α of* $\mathrm{Irr}(G)$ *the following hold*:

(a) $M(G, \alpha) = M(G_1, \alpha_1) \otimes M(G_2, \alpha_2) \otimes \cdots \otimes M(G_r, \alpha_r)$ *is the set of all block (G_1, α_1)-circulants in which the blocks are block (G_2, α_2)-circulants, in which the blocks are block (G_3, α_3)-circulants, etc.*

(b) $X(G, \alpha) = X(G_1, \alpha_1) \otimes X(G_2, \alpha_2) \otimes \cdots \otimes X(G_r, \alpha_r)$.

REMARK. Theorems on block (G, α)-circulants and on block matrices in which the blocks are (G, α)-circulants—theorems that are the analogues of Theorems 5.6.1, 5.6.2, 5.6.3, 5.6.4, 5.6.5, 5.7.1, 5.7.2, 5.7.3, and 5.7.4 of [7]—can be proved using the results of the first section. Since the analogues are straightforward extensions of the above results from [7], we will not state and prove them here.

3. EXAMPLES AND SPECIAL CASES

EXAMPLE 1 (Circulants). Let $G = C_n$, a cyclic group of order n. Let $G = \langle g \rangle$, namely, g is a generator of G. Let $\chi_1 = 1_G$, and χ_2 be the function $\chi_2 : G \to \mathbb{C}$ defined by $\chi_2(g^j) = \omega^j$, $0 \leqslant j \leqslant n-1$, where $\omega = e^{2\pi i/n}$. Write $\chi_3 = \chi_2^2$, $\chi_4 = \chi_2^3, \dots, \chi_n = \chi_2^{n-1}$. Then it is well known that $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \dots, \chi_n\}$. Denote by α this ordering of $\mathrm{Irr}(G)$. An arbitrary element of $\mathrm{cf}(G)$ has the form $\theta = \sum_{i=1}^n c_i \chi_i$, where the c_i's are arbitrary complex numbers. Note that $\chi_2^n = \chi_1$, so that

$$\theta\chi_j = \sum_{i=1}^{n} c_i \chi_i \chi_j = \sum_{i=1}^{n} c_i \chi_2^{i-1+j-1 \,(\mathrm{mod}\, n)}$$

$$= \sum_{i=1}^{n} c_i \chi_{i+j-1\,(\mathrm{mod}\, n)}.$$

Thus the $(j, i+j-1 \,(\mathrm{mod}\, n))$th entry of $M^\alpha(\theta)$ is c_i, and consequently

$$M^\alpha(\theta) = \begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_n \\ c_n & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_n & c_1 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_1 \end{pmatrix}.$$

We conclude that $M(G,\alpha)$ is the set of circulant matrices. Note that $M^\alpha(\chi_1)=I_n$, $M^\alpha(\chi_2)=\Pi=\Pi_n=\mathrm{circ}(0,1,0,\dots,0)$ (see [7] for notation), and $M^\alpha(\chi_i)=\Pi^{i-1}$ for $i=1,2,\dots,n$. Hence, an $n\times n$ matrix commutes with $M^\alpha(\chi_i)$ for all $i=1,2,\dots,n$ if and only if it commutes with Π.

Set $C_1=\{1\}$, $C_j=\{g^{j-1}\}$, $j=2,\dots,n$. Then the C_i, $1\leqslant i\leqslant n$, are the conjugacy class of G. Write $X=X(G,\alpha)$, then $X=\sqrt{n}\,F_n^*=\sqrt{n}\,F^*=(\omega^{(i-1)(j-1)})$ (see p. 32 of [7] for $F^*=F_n^*$). Next, note that

$$\theta=\sum_{i=1}^{n}c_i\chi_i=\sum_{i=1}^{n}c_i\chi_2^{i-1},$$

and by Corollary 1.3 we get that

$$\mathrm{circ}(c_1,c_2,\dots,c_n)=M^\alpha(\theta)=\sum_{i=1}^{n}c_i\Pi^{i-1}.$$

Also, by Theorem 1.4(d) we get that the eigenvalues of $\mathrm{circ}(c_1,c_2,\dots,c_n)$ are $\sum_{i=1}^{n}c_i\omega^{(i-1)(j-1)}$, $1\leqslant j\leqslant n$. These observations show that the following results of [7] are all special cases of our Corollary 1.3, Theorem 1.4, Theorem 1.5, Corollary 1.6, and Corollary 1.7: all the results of pp. 66, 67, 68, 72, 73; Theorem 3.2.4; Problems 9, 10 of p. 70; Problems 18, 19 of p. 81; Theorem 3.3.1 and its corollary; Results 3.4.10, 3.4.21; the corollaries of pp. 102 and 103; and others.

EXAMPLE 2 (Block circulant of an arbitrary level). Let $M=M(n_1,n_2,\dots,n_r)$ be the set of all block circulants with circulant blocks of level r and type $(n_1,n_2,\dots,n_r)$. That is, M consists of all $n_1\times n_1$ block circulants in which the blocks are $n_2\times n_2$ block circulants in which the blocks are $n_3\times n_3$ block circulants, etc. (see p. 188 of [7]). If $G=C_{n_1}\times C_{n_2}\times\cdots\times C_{n_r}$, the direct product of the cyclic groups C_{n_i} (of order n_i), then by Corollary 2.4 and Example 1 we have that $M(n_1,n_2,\dots,n_r)=M(G,\alpha)$ and $X(G,\alpha)=\sqrt{n_1n_2\dots n_r}\,F_{n_1}^*\otimes F_{n_2}^*\otimes\cdots\otimes F_{n_r}^*$. Here α is the ordering obtained by Corollary 2.4 (and can be explicitly written down using the proof of Theorem 2.1). For each $i=n_1,n_2,\dots,n_r$ and $j=1,2,\dots,n_i$ let $T_{ij}=I_{n_1}\otimes I_{n_2}\otimes\cdots\otimes I_{n_{i-1}}\otimes(\Pi_{n_i})^j\otimes I_{n_{i+1}}\otimes\cdots\otimes I_{n_r}$. Set $T_i=T_{i1}$ for $i=n_1,n_2,\dots,n_r$. Clearly $(T_i)^j=T_{ij}$ for $1\leqslant j\leqslant n_i$. If $\chi\in\mathrm{Irr}(G)$, then $M^\alpha(\chi)$ is a product of some T_{ij}'s. These observations show that Theorems 5.8.1, 5.8.2, 5.8.3, 5.8.4, and 5.8.5 of [7] are special cases of the results of Sections 1 and 2. In fact the following (known) corollary is a special case of Sections 1 and 2.

COROLLARY 3.1. *Let* $M = M(n_1, n_2, \dots, n_r)$, $F_{n_i}^*$, Π_{n_i}, *and* T_i, $1 \leqslant i \leqslant n$, *be as above. Then*:

(a) *If* $A, B \in M$ *and* $a, b \in \mathbb{C}$, *then* A^t, A^*, $aA + bB$, AB, $A^{\div}$, *and* A^k *for* $k \geqslant 0$ *are all in* M.

(b) *The elements of* M *commute, and they are simultaneously diagonalized by the matrix* $E = F_{n_1}^* \otimes F_{n_2}^* \otimes \cdots \otimes F_{n_r}^*$.

(c) *The following are equivalent for a square matrix* A *of order* $n = n_1 n_2 \cdots n_r$:

  (i) $A \in M$.
  (ii) $A = E^* \Delta E$, *where* E *is as in* (b), *and* Δ *an arbitrary* $n \times n$ *diagonal matrix over* $\mathbb{C}$.
  (iii) A *is a polynomial* (*of* r *variables*) *in the* T_i*'s*, $i = n_1, n_2, \dots, n_r$.
  (iv) A *commutes with the* T_i*'s*, $i = n_1, n_2, \dots, n_r$.

(d) *For each* l, *the eigenvalues of* T_l *are* $w_{n_l}^j$, $j = 1, 2, \dots, n_l$, *where* $w_{n_l} = \exp(2\pi i / n_l)$. *The eigenvalues of an arbitrary member of* M *can be computed using this part* (c) (iii).

EXAMPLE 3. The set of (G, α)-circulants can be computed for any group G for which the character table is known. Then new examples can be found using "direct products" of the old examples. Here is a demonstration. The smallest nonableian group is S_3, the symmetric group on 3 letters. Its character table [with respect to some ordering α of $\mathrm{Irr}(S_3)$] is

$$X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & -1 \end{pmatrix}.$$

(See p. 50 of [9].) The first row of X contains the values of χ_1, the second of χ_2, and the third of χ_3. We see that $\chi_1\chi_1 = \chi_1$, $\chi_1\chi_2 = \chi_2$, $\chi_1\chi_3 = \chi_3$, $\chi_2\chi_2 = \chi_1$, $\chi_2\chi_3 = \chi_3$, and $\chi_3^2 = \chi_1 + \chi_2 + \chi_3$. An arbitrary class function θ of S_3 has the form $\theta = a\chi_1 + b\chi_2 + c\chi_3$, where $a, b, c \in \mathbb{C}$. Therefore, $\theta\chi_1 = a\chi_1 + b\chi_2 + c\chi_3$, $\theta\chi_2 = b\chi_1 + a\chi_2 + c\chi_3$, $\theta\chi_3 = c\chi_1 + c\chi_2 + (a + b + c)\chi_3$. Hence, the set of (S_3, α)-circulants is the set of all matrices of the form

$$\begin{pmatrix} a & b & c \\ b & a & c \\ c & c & a+b+c \end{pmatrix}.$$

The matrices here are symmetric, since X is real and $X^*DX = I$ for a diagonal D.

Let $G = C_2 \times S_3 \times S_3 \times C_3$. Then for some α', the set of (G, α')-circulants (using Corollary 2.4) is the set of block matrices of the form

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

where A and B are block matrices of the form

$$\begin{pmatrix} C & D & E \\ D & C & E \\ E & E & C+D+E \end{pmatrix},$$

where C, D, E are block matrices of the form

$$\begin{pmatrix} K & L & M \\ L & K & M \\ M & M & K+L+M \end{pmatrix},$$

where K, L, and M are 3×3 circulants. Also, $X(G, \alpha') = \sqrt{6}\, F_2^* \otimes X \otimes X \otimes F_3^*$. This $X(G, \alpha')$ simultaneously diagonalizes all the (G, α')-circulants. Many other examples can be given by varying the groups and direct-multiplying them.

4. CHARACTERS AND NONNEGATIVE MATRICES

Let χ be a character of the finite group G; then $[\chi\chi_i, \chi_j]$ are all nonnegative integers where $\chi_i, \chi_j \in \mathrm{Irr}(G)$. It follows that the matrix $M^\alpha(\chi)$ is a nonnegative matrix with integer entries (by a nonnegative matrix we mean a matrix with nonnegative entries). Then results on nonnegative matrices can be used to find further connections between χ and $M^\alpha(\chi)$, which in turn enable us to obtain results (old and new) on group characters as consequences of theorems on nonnegative matrices. We give here one proposition that is used in obtaining such applications; other details and proofs can be found in [1], [5], and [6]. Our terminology for nonnegative matrices (irreducible, primitive, etc.) is taken from the second chapter of [2]. For the notation from character theory [faithful, $Z(\chi)$ etc.] see [8, Chapter 2].

PROPOSITION 4.1. *Let G be a finite group and α an ordering of* Irr(G). *Let χ be a character of G. Then in addition to the properties of $M^{\alpha}(\chi)$ described in Section* 1, *we have*:

(a) $M^{\alpha}(\chi)$ *is a nonnegative matrix with integer entries. Its leading (Perron-Frobenius) eigenvalue is* $\chi(1)$, *and a positive eigenvector corresponding to it is* $(\chi_1(1), \chi_2(1), \dots, \chi_k(1))^t$, *where* $Irr(G) = \{\chi_1, \chi_2, \dots, \chi_k\}$.

(b) χ *is faithful if and only if* $M^{\alpha}(\chi)$ *is* Irr*educible.*

(c) $Z(\chi) = 1$ *if and only if* $M^{\alpha}(\chi)$ *is primitive. In this case the primitivity index of* $M^{\alpha}(\chi)$ *is equal to the so-called character covering number of* χ (*see* [1] *for definition*).

Using this proposition, results on powers of characters and on values of character can be obtained as special cases of theorems on nonnegative matrices (see [1], [5], and [6]). So the mapping $\theta \to M^{\alpha}(\theta)$ can be useful for both characters and matrices.

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